

NEW HARDY-TYPE INEQUALITIES VIA OPIAL INEQUALITIES

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ABSTRACT. In this paper, we will prove several new inequalities of Hardy's type with explicit constants. The main results will be proved by making use of some generalizations of Opial's type inequalities and Hölder's inequality. To the best of the author's knowledge this method has not been used before in investigation of this type of inequalities. From these inequalities one can establish some new inequalities of differential forms which are inequalities of Wirtinger's type.

1. INTRODUCTION

The classical Hardy inequality (see [13]) states that for $f \geq 0$ and integrable over any finite interval $(0, x)$ and f^p is integrable and convergent over $(0, \infty)$ and $p > 1$, then

$$(1.1) \quad \int_0^\infty \left(\frac{1}{x} \left(\int_0^x f(t) dt \right) dx \right)^p \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx,$$

unless $f = 0$. The constant $(p/(p-1))^p$ is the best possible. This inequality has been proved by Hardy in 1925, but it has been appeared as the continuous version of a discrete inequality in his work in 1920 when he aimed to find a new elementary proof of Hilbert's inequality for double series. The discrete version of (1.1) is given by (see [14]) the discrete inequality

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n, \quad (a_n > 0, \quad p > 1).$$

During the past decades, the study of Hardy inequalities (continuous and discrete) or Hardy operators focused on the investigations of new inequalities or operators with weighted functions. These results are of interest and important in analysis not only because the mappings are optimal in the sense that the size of weight classes cannot be improved, but also because the weight conditions themselves are interest. This intensively investigated area of mathematical analysis resulted in the publication of numerous research papers and books, we refer the reader to the books [26, 18, 19] and the papers [5, 6, 16, 17]. The inequality (1.1) was

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extended to the form

$$(1.2) \quad \int_a^b \left(\left(\int_a^x f(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p(x) v(x) dx \right)^{\frac{1}{q}},$$

with a, b real numbers, satisfying $a < b$, and u, v are positive and measurable functions in the interval (a, b) , p, q are positive parameters satisfying $0 < q < \infty$ and $1 \leq p < \infty$. The main idea in this type of inequalities is to find the optimal value of the constant C (see [26]). Another type of inequalities of Hardy's type has been proved by Beesack in [6]. In particular, Beesack proved some inequalities of the form

$$(1.3) \quad \int_a^b s(x) \left(\int_0^x f(t) dt \right)^p dx \leq \int_a^b r(x) f^p(x) dx,$$

where the weighted functions r and s satisfy the Euler-Lagrange differential equation

$$(1.4) \quad \frac{d}{dx}(r(x)(y')^{p-1}) + s(x)y^{p-1}(x) = 0.$$

The method of the proofs in [6] depends on the existence of positive solution y of (1.4) which satisfies $y > 0$ and $y' > 0$ on the interval (a, b) . The idea of the present paper has been appeared during my work in some papers deal with disconjugacy and gaps between zeros of some differential equations ([28, 29, 3]), which require some inequalities of the form

$$(1.5) \quad \int_a^b s(x)y^p dx \leq C \int_a^b r(x)(y'(x))^p dx,$$

where C is the constant of the inequality depends on the weighted functions and needs to be determined. The differential forms of Hardy's inequalities (like 1.5) are of Wirtinger's type (see [1, 30]). This type of inequalities are useful in the analysis of qualitative behavior of solutions of differential equations and can be used in investigations of disconjugacy and disfocality of solutions (see [28, 29, 3]).

Kaijser et al. [17] pointed out that the inequality (1.1) is just a special case of the much more general (Hardy-Knopp type) inequality

$$(1.6) \quad \int_0^\infty \Phi \left(\frac{1}{x} \int_0^x f(t) dt \right) \frac{dx}{x} \leq \int_0^\infty \Phi(f(x)) \frac{dx}{x},$$

where Φ is a convex function on $(0, \infty)$. Recently the inequality (1.6) has been generalized by Kaijser et al. [16] by using the properties of convex functions and by employed Jensen's inequality and Fubini's Theorem.

Our aim in this paper is to prove some inequalities with weighted functions of Hardy's type by making use of some generalizations of Opial inequalities and Hölder's inequality. The method that, we will apply in this paper is different from the techniques used in the above mentioned papers and books. Throughout the paper some special cases are derived. The differential forms of these inequalities

which are inequalities of Wirtinger's type can be obtained and due to the limited space the details will be left to the interested reader.

2. MAIN RESULTS

In this section, we will prove the main results by making use of the Hölder inequality

$$(2.1) \quad \int_a^b |f(x)g(x)| dx \leq \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}} \left[\int_a^b |g(x)|^q dx \right]^{\frac{1}{q}},$$

where $a, b \in \mathbb{I}$ and $f; g \in C(\mathbb{I}, \mathbb{R})$, $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and some generalizations of Opial's inequality will be stated latter. The Opial inequality states that (see [25]): If y is absolutely continuous on $[a, b]$ and $y(a) = y(b) = 0$, then

$$(2.2) \quad \int_a^b |y(x)| |y'(x)| dx \leq \frac{(b-a)}{4} \int_a^b |y'(x)|^2 dx,$$

with a best constant $1/4$. Since the discovery of Opial inequality much work has been done, and many papers which deal with new proofs, various generalizations and extensions have appeared in the literature. In further simplifying the proof of the Opial inequality which had already been simplified by Olech [24], Beesack [7], Levinson [21], Mallows [22] and Pederson [27], it is proved that if y is real absolutely continuous on (a, b) and with $y(a) = 0$, then

$$(2.3) \quad \int_a^b |y(x)| |y'(x)| dx \leq \frac{b}{2} \int_a^b |y'(x)|^2 dx.$$

These inequalities and their extensions and generalizations are the most important and fundamental inequalities in the analysis of qualitative properties of solutions of different types of differential equations, (see [28, 29, 3]).

Throughout the paper, all functions are assumed to be positive and measurable and all the integrals appear in the inequalities are exist and finite.

Theorem 2.1. *Assume that r, s be positive and continuous functions on the interval (a, b) . Then*

$$(2.4) \quad \left(\int_a^b r(x) \left(\int_a^x f(t) dt \right) dx \right)^2 \leq \int_a^b \frac{R^2(x, b)}{s(x)} dx \left(\int_a^b s(x) (f(x))^2 dx \right),$$

for all functions $f \geq 0$, where

$$R(x, b) = \int_x^b r(x) dx.$$

Proof. Let $F(x) = \int_a^x f(t) dt$. It is clear that $F(a) = 0$ and $F'(x) = f(x)$. This gives us that

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right) dx = \int_a^b r(x) F(x) dx.$$

Integrating by parts the right hand side, we have

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right) dx = -R(x, b)F(x)|_a^b + \int_a^b R(x, b)F'(x) dx.$$

Using the assumptions $R(b, b) = 0$ and $F(a) = 0$, we have

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right) dx &= \int_a^b R(x, b)F'(x) dx \\ &= \int_a^b \frac{R(x, b)}{\sqrt{s(x)}} \sqrt{s(x)} F'(x) dx. \end{aligned}$$

Applying the inequality (2.1) with $p = q = 2$, we see that

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right) dx &\leq \left(\int_a^b \frac{R^2(x, b)}{s(x)} dx \right)^{\frac{1}{2}} \left(\int_a^b s(x) (F'(x))^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_a^b \frac{R^2(x, b)}{s(x)} dx \right)^{\frac{1}{2}} \left(\int_a^b s(x) (f(x))^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

which is the desired inequality (2.4). The proof is complete.

As in the proof of Theorem 2.1, by putting $F(x) = \int_x^b f(t) dt$, we can prove the following result.

Theorem 2.2. *Assume that r, s be positive and continuous functions on the interval (a, b) . Then*

$$(2.5) \quad \left(\int_a^b r(x) \left(\int_x^b f(t) dt \right) dx \right)^2 \leq \int_a^b \frac{R^2(a, x)}{s(x)} dx \left(\int_a^b s(x) (f(x))^2 dx \right),$$

for all functions $f \geq 0$, where

$$R(a, x) = \int_a^x r(x) dx.$$

In the following, we apply the inequality (2.3) to prove a new inequality of Hardy's type.

Theorem 2.3. *Assume that r be a positive and continuous function on the interval (a, b) . Then*

$$(2.6) \quad \int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx \leq (b-a) \sup_{a < x < b} \left(\int_x^b r(x) dx \right) \int_a^b (f(x))^2 dx,$$

for all functions $f \geq 0$.

Proof. We proceed as in the proof of Theorem 2.1 to get

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx = \int_a^b r(x) F^2(x) dx.$$

Integrating by parts the right hand side, we have

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx = -R(x, b)F^2(x)|_a^b + 2 \int_a^b R(x, b)F(x)F'(x) dx.$$

Using the assumptions $R(b, b) = 0$ and $F(a) = 0$, we have

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx &= 2 \int_a^b R(x, b) F(x) F'(x) dx \\ &\leq 2 \sup_{a < x < b} R(x, b) \int_a^b F(x) F'(x) dx. \end{aligned}$$

Applying the inequality (2.3), since $F(a) = 0$, we obtain

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx &\leq (b-a) \sup_{a < x < b} R(x, b) \int_a^b (F'(x))^2 dx \\ &= (b-a) \sup_{a < x < b} R(x, b) \left(\int_a^b (f(x))^2 dx \right), \end{aligned}$$

which is the desired inequality (2.6). The proof is complete.

As in the proof of Theorem 2.3, one can prove the following result.

Theorem 2.4. *Assume that r be positive and continuous function on the interval (a, b) . Then*

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^2 dx \leq (b-a) \sup_{a < x < b} \left(\int_a^x r(x) dx \right) \int_a^b (f(x))^2 dx,$$

for all functions $f \geq 0$.

In the following, we will apply a generalization of (2.3) due to Beesack [7] to prove a new inequality of Hardy's type. The inequality due to Beesack states that: If y is an absolutely continuous function on $[a, b]$ with $y(a) = 0$ (or $y(b) = 0$) then

$$(2.7) \quad \int_a^b |y(t)| |y'(t)| dt \leq \frac{1}{2} \int_a^b \frac{1}{r(t)} dt \int_a^b r(t) |y'(t)|^2 dt,$$

where $r(t)$ is positive and continuous function with $\int_a^b dt/r(t) < \infty$. Using this inequality on the term

$$2 \int_a^b R(x, b) F(x) F'(x) dx \leq 2 \sup_{a < x < b} R(x, b) \int_a^b F(x) F'(x) dx,$$

we see that

$$\begin{aligned} 2 \int_a^b R(x, b) F(x) F'(x) dx &\leq 2 \sup_{a < x < b} R(x, b) \frac{1}{2} \left(\int_a^b \frac{dx}{s(x)} \right) \int_a^b s(x) (F'(x))^2 dx \\ &= \sup_{a < x < b} R(x, b) \left(\int_a^b \frac{dx}{s(x)} \right) \int_a^b s(x) (f(x))^2 dx. \end{aligned}$$

Using this inequality and proceeding as in the proof of Theorem 2.3, we have the following inequality.

Theorem 2.5. Assume that r, s be positive and continuous functions in the interval (a, b) . Then

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx \leq \sup_{a < x < b} R(x, b) \left(\int_a^b \frac{dx}{s(x)} \right) \int_a^b s(x) (f(x))^2 dx,$$

for all functions $f \geq 0$.

Theorem 2.6. Assume that r, s be positive and continuous functions on the interval (a, b) . Then

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^2 dx \leq \sup_{a < x < b} R(a, x) \left(\int_a^b \frac{dx}{s(x)} \right) \int_a^b s(x) (f(x))^2 dx,$$

for all functions $f \geq 0$.

In the following, we apply the Maroni inequality [23] which states that: If y is an absolutely continuous function on $[a, b]$ with $y(a) =$ (or $y(b) = 0$), then

$$(2.8) \quad \int_a^b |y(t)| |y'(t)| dt \leq \frac{1}{2} \left(\int_a^b \left(\frac{1}{r(t)} \right)^{p-1} dt \right)^{\frac{2}{p}} \left(\int_a^b r(t) |y'(t)|^q dt \right)^{\frac{2}{q}},$$

where $\int_a^b (1/r(t))^{p-1} dt < \infty$, $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Applying the inequality (2.8) on the term $\sup_{a < x < b} R(x, b) \int_a^b F(x) F'(x) dx$, we see that

$$\begin{aligned} 2 \int_a^b R(x, b) F(x) F'(x) dx &\leq 2 \sup_{a < x < b} R(x, b) \int_a^b F(x) F'(x) dx \\ &\leq \sup_{a < x < b} R(x, b) \left(\int_a^b \left(\frac{1}{s(x)} \right)^{p-1} dx \right)^{\frac{2}{p}} \left(\int_a^b s(x) (F'(x))^q dx \right)^{\frac{2}{q}} \\ &= \sup_{a < x < b} R(x, b) \left(\int_a^b \left(\frac{1}{s(x)} \right)^{p-1} dx \right)^{\frac{2}{p}} \left(\int_a^b s(x) (f(x))^q dx \right)^{\frac{2}{q}}. \end{aligned}$$

Using this inequality and proceed as in the proof of Theorem 2.3, we have the following results.

Theorem 2.7. Assume that r, s be positive and continuous functions in the interval (a, b) and $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx &\leq \sup_{a < x < b} R(x, b) \left(\int_a^b \left(\frac{1}{s(x)} \right)^{p-1} dx \right)^{\frac{2}{p}} \\ &\quad \times \left(\int_a^b s(x) (f(x))^q dx \right)^{\frac{2}{q}}, \end{aligned}$$

for all functions $f \geq 0$.

Theorem 2.8. Assume that r be a positive function in the interval (a, b) such that $\int_a^b (1/R(a, x))^{p-1} dx < \infty$, $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} \int_a^b r(x) \left(\int_x^b f(t) dt \right)^2 dx &\leq \sup_{a < x < b} R(a, x) \left(\int_a^b \left(\frac{1}{s(x)} \right)^{p-1} dx \right)^{\frac{2}{p}} \\ &\quad \times \left(\int_a^b s(x) (f(x))^q dx \right)^{\frac{2}{q}} \end{aligned}$$

for all functions $f \geq 0$.

Yang [32] simplified the Beesack proof and extended the inequality (2.7) and proved that: If y is an absolutely continuous function on $[a, b]$ with $y(a) = 0$ (or $y(b) = 0$) then

$$(2.9) \quad \int_a^b q(t) |y(t)| |y'(t)| dt \leq \frac{1}{2} \int_a^b \frac{1}{r(t)} dt \int_a^b r(t) q(t) |y'(t)|^2 dt,$$

where $r(t)$ is a positive and continuous function with $\int_a^b dt/r(t) < \infty$ and $q(t)$ is a positive, bounded and non-increasing function on $[a, b]$. Applying the inequality (2.9) on the term $\int_a^b R(x, b) F(x) F'(x) dx$, we see that

$$\begin{aligned} &\int_a^b R(x, b) F(x) F'(x) dx \\ &\leq \frac{1}{2} \left(\int_a^b \frac{1}{s(x)} dx \right) \int_a^b R(x, b) s(x) (F'(x))^2 dx \\ &= \frac{1}{2} \left(\int_a^b \frac{1}{s(x)} dx \right) \int_a^b R(x, b) s(x) (f(x))^2 dx. \end{aligned}$$

This gives us the following results.

Theorem 2.9. Assume that r, s be positive functions in the interval (a, b) such that $\int_a^b dt/s(t) < \infty$ and $R(x, b)$ is a positive, bounded and non-increasing function on $[a, b]$. Then

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^2 dx \leq \left(\int_a^b \frac{dx}{s(x)} \right) \int_a^b R(x, b) s(x) (f(x))^2 dx,$$

for all functions $f \geq 0$.

Theorem 2.10. Assume that r, s be positive and continuous functions on the interval (a, b) such that $\int_a^b dt/s(t) < \infty$ and $R(a, x)$ is a positive, bounded and non-increasing function on $[a, b]$. Then

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^2 dx \leq \left(\int_a^b \frac{dx}{s(x)} \right) \left(\int_a^b R(a, x) s(x) (f(x))^2 dx \right),$$

for all functions $f \geq 0$.

In the following, we will apply the inequality due to Hua [15] to prove a new inequality of Hardy's type. The inequality due to Hua states that: If y is an

absolutely continuous function with $y(a) = 0$ (or $y(b) = 0$), then

$$(2.10) \quad \int_a^b |y(t)|^p |y'(t)| dt \leq \frac{(b-a)^p}{p+1} \int_a^b |y'(t)|^{p+1} dt,$$

where p is a positive integer.

Theorem 2.11. *Assume that r be positive and continuous function on the interval (a, b) and p is a positive integer. Then*

$$(2.11) \quad \int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq (b-a)^p \sup_{a < x < b} \left(\int_x^b r(t) dt \right) \int_a^b (f(x))^{p+1} dx,$$

for all functions $f \geq 0$.

Proof. Let $F(x) = \int_a^x f(t) dt$. It is clear that $F(a) = 0$ and $F'(x) = f(x) > 0$. This gives that

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx = \int_a^b r(x) F^{p+1}(x) dx.$$

Integrating by parts the left hand side, we have

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx &= - R(x, b) F^{p+1}(x) \Big|_a^b \\ &\quad + (p+1) \int_a^b R(x, b) F^p(x) F'(x) dx. \end{aligned}$$

Using the assumptions $R(b, b) = 0$ and $F(a) = 0$, we have

$$(2.12) \quad \begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx &= (p+1) \int_a^b R(x, b) F^p(x) F'(x) dx \\ &\leq (p+1) \sup_{a < x < b} R(x, b) \int_a^b F^p(x) F'(x) dx. \end{aligned}$$

Applying the inequality (2.10), since $F(a) = 0$, on the term $\int_a^b F^p(x) F'(x) dx$, we have

$$(2.13) \quad \int_a^b F^p(x) F'(x) dx \leq \frac{(b-a)^p}{p+1} \int_a^b (F'(x))^{p+1} dx.$$

Substituting (2.12) into (2.13), we have that

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq (b-a)^p \sup_{a < x < b} R(x, b) \int_a^b (f(x))^{p+1} dx,$$

which is the desired inequality (2.11). The proof is complete.

As in the proof of Theorem 2.11 one can prove the following result.

Theorem 2.12. *Assume that r be positive and continuous function on the interval (a, b) and p is a positive integer. Then*

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \leq (b-a)^p \sup_{a < x < b} \left(\int_a^x r(t) dt \right) \int_a^b (f(x))^{p+1} dx,$$

for all functions $f \geq 0$.

Boyd and Wong [11] extended the inequality (2.10) for general values of $p > 0$ and proved that if y is an absolutely continuous function on $[a, b]$ with $y(a) = 0$ (or $y(b) = 0$), then

$$(2.14) \quad \int_a^b s(t) |y(t)|^p |y'(t)| dt \leq \frac{1}{\lambda_0(p+1)} \int_a^b r(t) |y'(t)|^{p+1} dt,$$

where r and s are nonnegative functions in $C^1[a, b]$, λ_0 is the smallest eigenvalue of the boundary value problem

$$(r(t) (u'(t))^p)' = \lambda s'(t) u^p(t),$$

with $u(a) = 0$ and $u(b) = 0$ for which $u' > 0$ in $[a, b]$. Applying the inequality (2.14) on the term $(p+1) \int_a^b R(x, b) F^p(x) F'(x) dx$, we have

$$(2.15) \quad (p+1) \int_a^b R(x, b) F^p(x) F'(x) dx \leq \frac{1}{\lambda_0} \int_a^b s(t) (F'(x))^{p+1} dx,$$

where r and s are nonnegative functions in $C^1[a, b]$, and λ_0 is the smallest eigenvalue of the boundary value problem

$$(2.16) \quad (R(x, b) (u'(x))^p)' = \lambda s'(x) u^p(x),$$

where $R(x, b) = \int_x^b r(t) dt$, with $u(a) = 0$ and $u(b) = 0$ for which $u' > 0$ in $[a, b]$. Using (2.15) and proceeding as in the proof of Theorem 2.11, we have the following result.

Theorem 2.13. *Assume that r, s be positive and continuous functions on (a, b) and p is a positive integer. Then*

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq \frac{1}{\lambda_0} \int_a^b s(t) (f(x))^{p+1} dx,$$

for all functions $f \geq 0$ where λ_0 is the smallest eigenvalue of the boundary value problem (2.16).

In the following, we will apply the Calvert inequality (see [12])

$$(2.17) \quad \int_a^b |y(t)|^p |y'(t)| dt \leq \frac{1}{p+1} \left(\int_a^b s^{\frac{-1}{p}}(t) dt \right)^p \int_a^b s(t) |y'(t)|^{p+1} dt,$$

where $y(a) = 0$ (or $y(b) = 0$) and $s(t) > 0$, to prove a new inequality of Hardy's type. Applying the inequality (2.17) on the term $\int_a^b F^p(x) F'(x) dx$, we have

$$\int_a^b (F(x))^p F'(x) dx \leq \frac{1}{p+1} \left(\int_a^b s^{\frac{-1}{p}}(x) dx \right)^p \int_a^b s(x) (F'(x))^{p+1} dx.$$

Substituting this inequality into (2.12) and proceeding as in the proof of Theorem 2.11, we have the following results.

Theorem 2.14. Assume that r, s be positive and continuous functions in (a, b) and p is a positive integer. Then

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq \int_a^b s(x) (f(x))^{p+1} dx,$$

for all functions $f \geq 0$, where $C = \sup_{a < x < b} \left(\int_x^b r(t) dt \right) \left(\int_a^b s^{\frac{-1}{p}}(x) dx \right)^p$.

Theorem 2.15. Assume that r, s be positive and continuous functions on (a, b) and p is a positive integer. Then

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} \leq C \int_a^b s(x) (f(x))^{p+1} dx,$$

for all functions $f \geq 0$, where $C = \sup_{a < x < b} \left(\int_a^x r(t) dt \right) \left(\int_a^b s^{\frac{-1}{p}}(x) dx \right)^p$.

In the following we apply the Yang inequality [32]

$$(2.18) \quad \int_a^b |y(t)|^p |y'(t)|^q dt \leq \frac{q}{p+q} (b-a)^p \int_a^b |y'(t)|^{p+q} dt.$$

where y is an absolutely continuous function on $[a, b]$ with $x(a) = 0$ (or $y(b) = 0$), $p \geq 0$ and $q \geq 1$, to prove a new inequality of Hardy's type. Applying the inequality (2.1) and the inequality (2.18) on the term

$$\int_a^b R(x, b) F^p(x) F'(x) dx,$$

we have

$$\begin{aligned} \int_a^b R(x, b) F^p(x) F'(x) dx &\leq \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}} \left(\int_a^b F^{pq}(x) (F'(x))^q dx \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{p+1} \right)^{\frac{1}{q}} (b-a)^p \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_a^b (F'(x))^{q(p+1)} dx \right)^{\frac{1}{q}}, \end{aligned}$$

where $1/p + 1/q = 1$. Using this inequality and proceeding as in the proof of Theorem 2.11, we get the following results.

Theorem 2.16. Assume that r be positive and continuous function on (a, b) and $p > 1$ is a positive integer. Then

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq C \left(\int_a^b (f(x))^{\frac{p(p+1)}{p-1}} dx \right)^{\frac{p-1}{p}},$$

for all functions $f \geq 0$, where $C = (p+1)^{\frac{1}{p}} (b-a)^p \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}}$.

Theorem 2.17. Assume that r be positive and continuous function on (a, b) and $p > 1$ is a positive integer. Then

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \leq \left(\int_a^b (f(x))^{\frac{p(p+1)}{p-1}} dx \right)^{\frac{p-1}{p}},$$

for all functions $f \geq 0$, where $C = (p+1)^{\frac{1}{p}} (b-a)^p \left(\int_a^b R^p(a, x) dx \right)^{\frac{1}{p}}$.

From Theorems 2.16, 2.17, we have the following inequality.

Corollary 2.1. Assume that r be positive and continuous function on (a, b) and $p > 1$ is a positive integer. Then

$$\left(\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \right)^{\frac{1}{p+1}} \leq C_1 \left(\int_a^b (f(x))^{\frac{p(p+1)}{p-1}} dx \right)^{\frac{p-1}{p(p+1)}},$$

for all functions $f \geq 0$, where

$$C = (p+1)^{\frac{1}{p(p+1)}} (b-a)^{\frac{p}{p+1}} \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p(p+1)}},$$

and

$$\left(\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \right)^{\frac{1}{p+1}} \leq C_2 \left(\int_a^b (f(x))^{\frac{p(p+1)}{p-1}} dx \right)^{\frac{p-1}{p(p+1)}},$$

for all functions $f \geq 0$, where

$$C = (p+1)^{\frac{1}{p(p+1)}} (b-a)^{\frac{p}{p+1}} \left(\int_a^b R^p(a, x) dx \right)^{\frac{1}{p(p+1)}}.$$

By choosing $q+1 = \frac{p(p+1)}{p-1}$, we have from Corollary 2.1, the following inequalities which are of the form (1.2).

Corollary 2.2. Assume that r be positive and continuous function on (a, b) and $p > 1$ is a positive integer. Then

$$\left(\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \right)^{\frac{1}{p+1}} \leq C_* \left(\int_a^b (f(x))^{q+1} dx \right)^{\frac{1}{q+1}},$$

for all functions $f \geq 0$, where

$$C_* = (p+1)^{\frac{1}{p(p+1)}} (b-a)^{\frac{p}{p+1}} \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p(p+1)}},$$

and

$$\left(\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \right)^{\frac{1}{p+1}} \leq C^* \left(\int_a^b (f(x))^{q+1} dx \right)^{\frac{1}{q+1}},$$

for all functions $f \geq 0$, where

$$C^* = (p+1)^{\frac{1}{p(p+1)}} (b-a)^{\frac{p}{p+1}} \left(\int_a^b R^p(a, x) dx \right)^{\frac{1}{p(p+1)}}.$$

Yang [33] extended the inequality (2.18) and proved that if $r(t)$ is a positive bounded function and y is an absolutely continuous on $[a, b]$ with $y(a) = 0$ (or $y(b) = 0$), $p \geq 0$, $q \geq 1$, then

$$(2.19) \quad \int_a^b r(t) |y(t)|^p |y'(t)|^q dt \leq \frac{q}{p+q} (b-a)^p \int_a^b r(t) |y'(t)|^{p+q} dt.$$

Applying the inequality (2.19) on the term $\int_a^b R(x, b) F^p(x) F'(x) dx$, we have

$$\begin{aligned} \int_a^b R(x, b) F^p(x) F'(x) dx &\leq \frac{1}{p+1} (b-a)^p \left(\int_a^b R(x, b) (F'(x))^{p+1} dx \right) \\ &= \frac{1}{p+1} (b-a)^p \left(\int_a^b R(x, b) (f(x))^{p+1} dx \right). \end{aligned}$$

Using this inequality and proceeding as in the proof of Theorem 2.11, we obtain the following results.

Theorem 2.18. Assume that r be positive and continuous function on (a, b) and $p \geq 0$ is a positive integer. Then

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq (b-a)^p \int_a^b R(x, b) (f(x))^{p+1} dx,$$

for all functions $f \geq 0$.

Theorem 2.19. Assume that r be positive and continuous function on (a, b) and $p \geq 0$ is a positive integer. Then

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \leq (b-a)^p \int_a^b R(a, x) (f(x))^{p+1} dx,$$

for all functions $f \geq 0$.

In the following, we apply an inequality due to Boyd [10] and the Hölder inequality to obtain new inequalities. The Boyd inequality states that: If $y \in C^1[a, b]$ with $y(a) = 0$ (or $y(b) = 0$), then

$$(2.20) \quad \int_a^b |y(t)|^\nu |y'(t)|^\eta dt \leq N(\nu, \eta, s) (b-a)^\nu \left(\int_a^b |y'(t)|^s dt \right)^{\frac{\nu+\eta}{s}},$$

where $\nu > 0$, $s > 1$, $0 \leq \eta < s$,

$$(2.21) \quad N(\nu, \eta, s) := \frac{(s-\eta) \nu^\nu \sigma^{\nu+\eta-s}}{(s-1)(\nu+\eta) (I(\nu, \eta, s))^\nu}, \quad \sigma := \left\{ \frac{\nu(s-1) + (s-\eta)}{(s-1)(\nu+\eta)} \right\}^{\frac{1}{s}},$$

and

$$I(\nu, \eta, s) := \int_0^1 \left\{ 1 + \frac{s(\eta-1)}{s-\eta} t \right\}^{-(\nu+\eta+s\nu)/s\nu} [1 + (\eta-1)t]^{1/\nu-1} dt.$$

Applying the inequality (2.1) and the inequality (2.20) on the term

$$\int_a^b R(x, b) F^p(x) F'(x) dx,$$

we have

$$\begin{aligned} \int_a^b R(x, b) F^p(x) F'(x) dx &\leq \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}} \left(\int_a^b F^{pq}(x) (F'(x))^q dx \right)^{\frac{1}{q}} \\ &\leq N^{\frac{1}{q}}(pq, q, s)(b-a)^p \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_a^b (F'(x))^s dx \right)^{\frac{p+1}{s}}, \end{aligned} \quad (2.22)$$

where $1/p + 1/q = 1$, and $N(pq, q, s)$ is determined from (2.21) by putting $\nu = pq$ and $\eta = q$. Using the inequality (2.22) and proceeding as in the proof of Theorem 2.11, we have the following results.

Theorem 2.20. *Assume that r be a positive functions in (a, b) , $p > 0$, $s > 1$, $0 \leq q < s$ and $1/p + 1/q = 1$. Then*

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq (p+1)C \left(\int_a^b (f(x))^s dx \right)^{\frac{p+1}{s}},$$

for all functions $f \geq 0$ where $C = N^{\frac{1}{q}}(pq, q, s)(b-a)^p \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}}$.

Theorem 2.21. *Assume that r be a positive and continuous function on (a, b) , $p > 1$, $1 < q < s$ and $1/p + 1/q = 1$. Then*

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \leq (p+1)C \left(\int_a^b (f(x))^s dx \right)^{\frac{p+1}{s}},$$

for all functions $f \geq 0$, where $C = N^{\frac{1}{q}}(pq, q, s)(b-a)^p \left(\int_a^b R^p(a, x) dx \right)^{\frac{1}{p}}$.

The inequality (2.20) has immediate application when $\eta = s$. In this case the inequality (2.20) becomes

$$(2.23) \quad \int_a^b |y(t)|^\nu \left| y'(t) \right|^\eta dt \leq L(\nu, \eta)(b-a)^\nu \left(\int_a^b \left| y'(t) \right|^\eta dt \right)^{\frac{\nu+\eta}{\eta}},$$

where

$$(2.24) \quad L(\nu, \eta) := \frac{\eta \nu^\eta}{\nu + \eta} \left(\frac{\nu}{\nu + \eta} \right)^{\frac{\nu}{\eta}} \left(\frac{\Gamma\left(\frac{\eta+1}{\eta} + \frac{1}{\nu}\right)}{\Gamma\left(\frac{\eta+1}{\eta}\right) \Gamma\left(\frac{1}{\nu}\right)} \right)^\nu,$$

and Γ is the Gamma function. Applying the inequality (2.23) on the term

$$\int_a^b F^{pq}(x) \left(F'(x) \right)^q dx,$$

we have

$$(2.25) \quad \int_a^b F^{pq}(x) \left(F'(x) \right)^q dx \leq L(pq, q)(b-a)^{pq} \left(\int_a^b \left(F'(x) \right)^q dx \right)^{\frac{pq+q}{q}}$$

where

$$(2.26) \quad L(pq, q) = \frac{(pq)^q}{p+1} \left(\frac{p}{p+1} \right)^p \left(\frac{\Gamma\left(\frac{q+1}{q} + \frac{1}{pq}\right)}{\Gamma\left(\frac{q+1}{q}\right) \Gamma\left(\frac{1}{pq}\right)} \right).$$

Using (2.25), we see that

$$\begin{aligned} \int_a^b R(x, b) F^p(x) F'(x) dx &\leq \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}} \left(\int_a^b F^{pq}(x) \left(F'(x) \right)^q dx \right)^{\frac{1}{q}} \\ &\leq L^{\frac{1}{q}}(pq, q)(b-a)^p \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_a^b \left(F'(x) \right)^q dx \right)^{p+1}, \end{aligned}$$

where $1/p + 1/q = 1$. This gives us the following results.

Theorem 2.22. Assume that r be positive and continuous function in (a, b) , $p, q > 1$ and $1/p + 1/q = 1$. Then

$$\int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx \leq (p+1)C \left(\int_a^b (f(x))^q dx \right)^{(p+1)},$$

for all functions $f \geq 0$, where $C = L^{\frac{1}{q}}(pq, q)(b-a)^p \left(\int_a^b R^p(x, b) dx \right)^{\frac{1}{p}}$ and $L^{\frac{1}{q}}(pq, q)$ is defined as in (2.26).

Theorem 2.23. Assume that r be positive and continuous function on (a, b) , $p, q > 1$ and $1/p + 1/q = 1$. Then

$$\int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx \leq (p+1)C \left(\int_a^b (f(x))^q dx \right)^{(p+1)},$$

for all functions $f \geq 0$, where $C = L^{\frac{1}{q}}(pq, q)(b-a)^p \left(\int_a^b R^p(a, x) dx \right)^{\frac{1}{p}}$ and $L^{\frac{1}{q}}(pq, q)$ is defined as in (2.26).

Next in the following, we will apply some Opial type inequalities obtained by Beesack and Das [8] (see also [31]) to prove some new inequalities of Hardy's types with two different weighted functions. These inequalities are presented in the following theorems.

Theorem 2.24. *Let p, q be positive real numbers such that $p+q > 1$, and let r, s be positive continuous functions in (a, b) such that $\int_a^b s^{\frac{-1}{p+q-1}}(t) dt < \infty$. If $y : [a, b] \rightarrow \mathbb{R}$ is delta differentiable with $y(a) = 0$, then*

$$(2.27) \quad \int_a^X r(x) |y(x)|^p |y'(x)|^q dx \leq K_1(a, X, p, q) \int_a^X s(x) |y'(x)|^{p+q} dx,$$

where

$$(2.28) \quad K_1(a, X, p, q) = \left(\frac{q}{p+q} \right)^{\frac{q}{p+q}} \times \left(\int_a^X (r(x))^{\frac{p+q}{p}} (s(x))^{-\frac{q}{p}} \left(\int_a^x s^{\frac{-1}{p+q-1}}(t) dt \right)^{(p+q-1)} dx \right)^{\frac{p}{p+q}}.$$

Theorem 2.25. *Assume that p, q be positive real numbers such that $p+q > 1$, and let r, s be nonnegative continuous functions in (a, b) such that $\int_b^a s^{\frac{-1}{p+q-1}}(t) dt < \infty$. If $y : [a, b] \rightarrow \mathbb{R}$ is delta differentiable with $y(b) = 0$, then we have*

$$(2.29) \quad \int_a^b r(x) |y(x)|^p |y'(x)|^q dx \leq K_2(X, b, p, q) \int_X^b s(x) |y'(x)|^{p+q} dx,$$

where

$$(2.30) \quad K_2(X, b, p, q) = \left(\frac{q}{p+q} \right)^{\frac{q}{p+q}} \times \left(\int_a^b (r(x))^{\frac{p+q}{p}} (s(x))^{-\frac{q}{p}} \left(\int_x^b s^{\frac{-1}{p+q-1}}(t) dt \right)^{(p+q-1)} dx \right)^{\frac{p}{p+q}}.$$

In the following, we assume that there exists $h \in (a, b)$ which is the unique solution of the equation

$$K(p, q) = K_1(a, h, p, q) = K_2(h, b, p, q) < \infty,$$

where $K_1(a, h, p, q)$ and $K_2(h, b, p, q)$ are defined as in Theorems 2.24 and 2.25. The combination of Theorems 2.24 and 2.5 gives the following result.

Theorem 2.26. *Let p, q be positive real numbers such that $pq > 0$ and $p+q > 1$, and let r, s be nonnegative continuous functions on (a, b) such that*

$\int_a^b s^{\frac{-1}{p+q-1}}(t)dt < \infty$. If $y : [a, b] \rightarrow \mathbb{R}$ is delta differentiable with $y(a) = 0 = y(b)$, then we have

$$(2.31) \quad \int_a^b r(x) |y(x)|^p |y'(x)|^q dx \leq K(p, q) \int_a^b s(x) |y'(x)|^{p+q} dx.$$

For $r = s$ in (2.27), we obtain the following special case from Theorem 2.24.

Corollary 2.3. *Let p, q be positive real numbers such that $p + q > 1$, and let r be a nonnegative continuous function in (a, b) such that $\int_a^b r^{\frac{-1}{p+q-1}}(t)dt < \infty$. If $y : [a, b] \rightarrow \mathbb{R}$ is delta differentiable with $y(a) = 0$, then we have*

$$(2.32) \quad \int_a^X r(x) |y(x)|^p |y'(x)|^q dx \leq K_1^*(a, b, p, q) \int_a^b r(x) |y'(x)|^{p+q} dx,$$

where

$$(2.33) \quad K_1^*(a, b, p, q) = \left(\frac{q}{p+q} \right)^{\frac{q}{p+q}} \times \left(\int_a^b r(x) \left(\int_a^x r^{\frac{-1}{p+q-1}}(t)dt \right)^{(p+q-1)} dx \right)^{\frac{p}{p+q}}.$$

Now, we apply Theorem 2.24 to prove a new inequality of Hardy's type. Using the fact that

$$\int_a^b R(x, b) F^p(x) F'(x) dx = \int_a^b \frac{R(x, b)}{r^{\frac{1}{q}}(x)} F^p(x) (r^{\frac{1}{q}}(x) F'(x)) dx$$

and applying the inequality (2.1), we have

$$(2.34) \quad \begin{aligned} \int_a^b \frac{R(x, b)}{r^{\frac{1}{q}}(x)} F^p(x) (r^{\frac{1}{q}}(x) F'(x)) dx &\leq \left(\int_a^b \frac{R^p(x, b)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ &\times \left(\int_a^b r(x) F^{pq}(x) (F'(x))^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Applying the inequality (2.27) on the term

$$\int_a^b r(x) F^{pq}(x) (F'(x))^q dx,$$

with $y = F$, and p is replaced by pq , note that $F(a) = 0$, we have that

$$(2.35) \quad \int_a^b r(x) F^{pq}(x) (F'(x))^q dx \leq K_1(a, b, pq, q) \int_a^b s(x) (F'(x))^{pq+q} dx,$$

where

$$(2.36) \quad \begin{aligned} K_1(a, b, pq, q) &= \left(\frac{q}{pq+q} \right)^{\frac{q}{pq+q}} \\ &\times \left(\int_a^b (r(x))^{\frac{pq+q}{p}} (s(x))^{-\frac{q}{pq}} \left(\int_a^x s^{\frac{-1}{pq+q-1}}(t)dt \right)^{(pq+q-1)} dx \right)^{\frac{pq}{pq+q}}. \end{aligned}$$

Substituting (2.35) into (2.34), we have that

$$\begin{aligned} \int_a^b \frac{R(x, b)}{r^{\frac{1}{q}}(x)} F^p(x) (r^{\frac{1}{q}}(x) F'(x)) dx &\leq K_1^{\frac{1}{q}}(a, X, pq, q) \left(\int_a^b \frac{R^p(x, b)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_a^b s(x) (F'(x))^{pq+q} dx \right)^{\frac{1}{q}}. \end{aligned}$$

This gives us the following result.

Theorem 2.27. *Let p, q be positive real numbers such that $p, q > 1$, and let r, s be nonnegative continuous functions on (a, b) such that $\int_a^b s^{\frac{-1}{p+q-1}}(t) dt < \infty$. Then*

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx &\leq (p+1) K_1^{\frac{1}{q}}(a, X, pq, q) \left(\int_a^b \frac{R^p(x, b)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ &\quad \left(\int_a^b s(x) (f(x))^{pq+q} dx \right)^{\frac{1}{q}}, \end{aligned}$$

for all functions $f \geq 0$, where $K_1(a, b, pq, q)$ is defined as in (2.36).

Applying Theorem 2.25 will give us the following result.

Theorem 2.28. *Let p, q be positive real numbers such that $p, q > 1$ and $1/p + 1/q = 1$, and let r, s be nonnegative continuous functions in (a, b) such that $\int_a^b s^{\frac{-1}{p+q-1}}(t) dt < \infty$. Then*

$$\begin{aligned} \int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx &\leq (p+1) K_2^{\frac{1}{q}}(a, X, pq, q) \left(\int_a^b \frac{R^p(a, x)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ &\quad \left(\int_a^b s(x) (f(x))^{pq+q} dx \right)^{\frac{1}{q}}, \end{aligned}$$

for all functions $f \geq 0$, where $K_2(a, b, pq, q)$ is defined by

$$\begin{aligned} K_2(a, b, pq, q) &= \left(\frac{q}{pq+q} \right)^{\frac{q}{pq+q}} \\ &\quad \times \left(\int_a^b (r(x))^{\frac{pq+q}{pq}} (s(x))^{-\frac{q}{pq}} \left(\int_x^b s^{\frac{-1}{pq+q-1}}(t) dt \right)^{(pq+q-1)} dx \right)^{\frac{pq}{pq+q}}. \end{aligned}$$

Remark 1. *One can apply Theorem 2.6 and Corollary 2.3 to obtain new inequalities of Hardy's type. The details are left to the interested reader.*

In the following, we will apply a new Opial type inequality due to Beesack [9] to prove new inequalities of Hardy's type. The inequality due to Beesack is given in the following theorem.

Theorem 2.29. *Let r, s be nonnegative, measurable functions on (α, τ) . Further assume that $k > 1$, $p > 0$, $0 < q < k$, and let $y(t)$ be absolutely continuous in (α, τ) such that $y(\alpha) = 0$. Then*

$$(2.37) \quad \int_{\alpha}^{\tau} r(t) |y(t)|^p |y'(t)|^q dt \leq K_1(p, q, k) \left[\int_{\alpha}^{\tau} s(t) |y'(t)|^k dt \right]^{(p+q)/k},$$

where

$$(2.38) \quad K_1(p, q, k) = \left(\frac{q}{q+p} \right)^{\frac{q}{k}} \times \left(\int_{\alpha}^{\tau} (r(y))^{\frac{k}{k-q}} (s(y))^{-\frac{q}{k-q}} \left(\int_a^y s^{\frac{-1}{k-1}}(t) dt \right)^{p(k-1)/(k-q)} dy \right)^{\frac{k-q}{k}}.$$

If instead of (α, τ) is replaced by (τ, β) and $y(\alpha) = 0$ is replaced by $y(\beta) = 0$, then

$$(2.39) \quad \int_{\tau}^{\beta} r(t) |y(t)|^p |y'(t)|^q dt \leq K_2(p, q, k) \left[\int_{\tau}^{\beta} s(t) |y'(t)|^k dt \right]^{(p+q)/k},$$

where

$$(2.40) \quad K_2(p, q, k) = \left(\frac{q}{q+p} \right)^{\frac{q}{k}} \times \left(\int_{\tau}^{\beta} (r(y))^{\frac{k}{k-q}} (s(y))^{-\frac{q}{k-q}} \left(\int_y^{\beta} s^{\frac{-1}{k-1}}(t) dt \right)^{p(k-1)/(k-q)} dy \right)^{\frac{k-q}{k}}.$$

Now, we apply inequality (2.37) and (2.39) to obtain new inequalities of Hardy's type. First we consider (2.37). Using the fact that

$$\int_a^b R(x, b) F^p(x) F'(x) dx = \int_a^b \frac{R(x, b)}{r^{\frac{1}{q}}(x)} F^p(x) (r^{\frac{1}{q}}(x) F'(x)) dx,$$

and applying the inequality (2.1), we have

$$(2.41) \quad \begin{aligned} \int_a^b \frac{R(x, b)}{r^{\frac{1}{q}}(x)} F^p(x) (r^{\frac{1}{q}}(x) F'(x)) dx &\leq \left(\int_a^b \frac{R^p(x, b)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ &\times \left(\int_a^b r(x) F^{pq}(x) (F'(x))^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Applying (2.37) on the term

$$\int_a^b r(x) F^{pq}(x) (F'(x))^q dx,$$

we have (note that $F(a) = 0$), that

$$(2.42) \quad \int_a^b r(x) F^{pq}(x) \left(F'(x)\right)^q dx \leq K_1(pq, q, k) \left[\int_a^b s(x) (F'(x))^k dx \right]^{(pq+q)/k},$$

where

$$(2.43) \quad K_1(pq, q, k) = \left(\frac{1}{1+p} \right)^{\frac{q}{k}} \times \left(\int_a^b (r(x))^{\frac{k}{k-q}} (s(x))^{-\frac{q}{k-q}} \left(\int_a^t s^{\frac{-1}{k-1}}(t) dt \right)^{pq(k-1)/(k-q)} dx \right)^{\frac{k-q}{k}}.$$

Substituting (2.42) into (2.41), we get that

$$\begin{aligned} & \int_a^b \frac{R(x, b)}{r^{\frac{1}{q}}(x)} F^p(x) (r^{\frac{1}{q}}(x) F'(x)) dx \\ & \leq K_1^{\frac{1}{q}}(pq, q, k) \left(\int_a^b \frac{R^p(x, b)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \times \left[\int_a^b s(x) (F'(x))^k dx \right]^{(p+1)/k}. \end{aligned}$$

This gives us the following results.

Theorem 2.30. *Let p, q be positive real numbers such that $p, q > 1$, and let r, s be nonnegative continuous functions on (a, b) . Then*

$$\begin{aligned} \int_a^b r(x) \left(\int_a^x f(t) dt \right)^{p+1} dx & \leq (p+1) K_1^{\frac{1}{q}}(pq, q, q) \left(\int_a^b \frac{R^p(x, b)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ & \times \left[\int_a^b s(x) (f(x))^k dx \right]^{(p+1)/k}, \end{aligned}$$

for all functions $f \geq 0$, where $K_1(pq, q, k)$ is defined as in (2.43).

Theorem 2.31. *Let p, q be positive real numbers such that $p, q > 1$, and let r, s be nonnegative continuous functions on (a, b) . Then*

$$\begin{aligned} \int_a^b r(x) \left(\int_x^b f(t) dt \right)^{p+1} dx & \leq (p+1) K_2^{\frac{1}{q}}(pq, q, q) \left(\int_a^b \frac{R^p(a, x)}{r^{\frac{1}{q}}(x)} dx \right)^{\frac{1}{p}} \\ & \times \left[\int_a^b s(x) (f(x))^k dx \right]^{(p+1)/k}, \end{aligned}$$

for all functions $f \geq 0$, where $K_2(pq, q, k)$ is defined by

$$\begin{aligned} K_2(pq, q, k) & = \left(\frac{1}{1+p} \right)^{\frac{q}{k}} \\ & \times \left(\int_a^b (r(x))^{\frac{k}{k-q}} (s(x))^{-\frac{q}{k-q}} \left(\int_t^b s^{\frac{-1}{k-1}}(t) dt \right)^{pq(k-1)/(k-q)} dx \right)^{\frac{k-q}{k}}. \end{aligned}$$

Remark 2. *The applications of the Opial inequality may be continue and one can obtain new inequalities of Hardy's type. On the other hand one can obtain the differential forms of the inequalities by replacing $\int_a^x f(t)dt$ by $g(x)$ on the right hand sides and replace $f(x)$ in the left hand sides by $g'(x)$. The details are left to the reader.*

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